Probability Theory

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Chapter 05: Continuous Random Variables

Continuous Random Variables

Examples of RVs whose set of possible values is uncountable

- The time that a train arrives at a specified stop
- The lifetime of a transistor

Definition

We say that X is a *continuous RV* if there exists a nonnegative function f, defined for all real $x \in (-\infty, \infty)$, having the property that, for any *measurable* set B of real numbers,

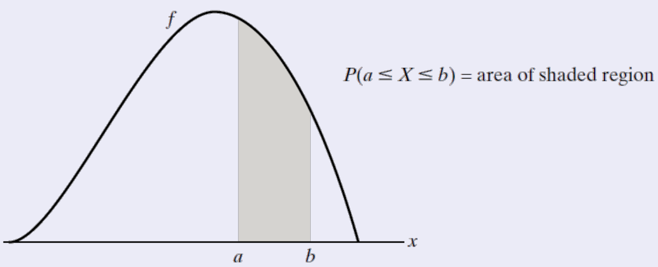
$$P\{X \in B\} = \int_B f(x)\ dx$$

The function f is called the *probability density function* of the random variable X.

$$1 = P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x)\ dx$$

Continuous Random Variables

Answering probability statements about continuous RVs



$$P\{a \le X \le b\} = \int_a^b f(x)\ dx \qquad P\{X = a\} = \int_a^a f(x)\ dx = 0$$

$$P\{a < X < b\} = P\{a \le X < b\} = P\{a < X \le b\} = P\{a \le X \le b\}$$

Continuous Random Variables

$f(a)$ is a measure of how likely it is that the RV will be near a

$$P\left\{a - \frac{\varepsilon}{2} \le X \le a + \frac{\varepsilon}{2}\right\} = \int_{a-\frac{\varepsilon}{2}}^{a+\frac{\varepsilon}{2}} f(x)\ dx \approx \varepsilon f(a)$$

when ε is small and when $f(\cdot)$ is continuous at $x = a$

Example

Find $P\{X > 1\}$, where X is a continuous RV whose pdf is given by

$$f(x) = \begin{cases} C(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$1 = \int_{-\infty}^{\infty} f(x)\ dx = C\int_0^2 4x - 2x^2\ dx = C\left(2x^2 - \frac{2x^3}{3}\right)\Big|_0^2 = \frac{8C}{3} \qquad \therefore C = \frac{3}{8}$$

$$P\{X > 1\} = \int_1^{\infty} f(x)\ dx = \frac{3}{8}\int_1^2 4x - 2x^2\ dx = \frac{3}{8}\left(2x^2 - \frac{2x^3}{3}\right)\Big|_1^2 = \frac{1}{2}$$

Continuous Random Variables

Cumulative Distribution Function

$$F(a) = P\{X \in (-\infty, a]\} = \int_{-\infty}^{a} f(x)\ dx$$

$$\frac{d}{da}F(a) = f(a)$$

Example

If X is continuous with distribution function F_X and density function f_X, find the density function of $Y = X^2$

$$\begin{aligned} F_Y(a) &= P\{Y \le a\} \\ &= P\{X^2 \le a\} \\ &= P\{-\sqrt{a} \le X \le \sqrt{a}\} \\ &= F_X(\sqrt{a}) - F_X(-\sqrt{a}) \end{aligned}$$

$$\begin{aligned} f_Y(a) &= \tfrac{d}{da}F_Y(a) \\ &= \tfrac{d}{da}\left(F_X(\sqrt{a}) - F_X(-\sqrt{a})\right) \\ &= \frac{1}{2\sqrt{a}}\left(f_X(\sqrt{a}) + f_X(-\sqrt{a})\right) \end{aligned}$$

Expectation of Continuous RVs

The expectation (or expected value) of a continuous RV X

If X is a continuous RV having pdf $f(x)$, then,

$$f(x)\ dx \approx P\{x \le X \le x + dx\} \qquad \text{for } dx \text{ small}$$

it is easy to see that

$$E[X] = \int_{-\infty}^{\infty} x f(x)\ dx$$

Example

Find $E[X]$ when the pdf of X is: $f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f(x)\ dx \\ &= \int_0^1 2x^2\ dx = \frac{2}{3} \end{aligned}$$

Expectation of Continuous RVs

Example

Find $E[e^X]$ when the pdf of X is: $f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

Let $Y = e^X$. We start by determining F_Y and then f_Y. Now, for $1 \le y \le e$,

$$\begin{aligned} F_Y(y) &= P\{Y \le y\} \\ &= P\{e^X \le y\} \\ &= P\{X \le \log(y)\} \\ &= \int_{-\infty}^{\log(y)} f(x)\ dx \\ &= \int_0^{\log(y)} 1\ dx \\ &= \log(y) \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \tfrac{d}{dy} F_Y(y) \\ &= \frac{1}{y} \qquad (1 \le y \le e) \\ E[e^X] &= E[Y] \\ &= \int_{-\infty}^{\infty} y f_Y(y)\ dy \\ &= \int_1^e 1\ dy \\ &= e - 1 \end{aligned}$$

Expectation of a Function of a Continuous RV

Proposition

If X is a continuous RV having pdf $f(x)$, then, for any real-valued function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\ dx$$

Revisiting the previous example

Find $E[e^X]$ when the pdf of X is: $f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} E[e^X] &= \int_{-\infty}^{\infty} e^x f(x)\ dx \\ &= \int_0^1 e^x\ dx \\ &= e - 1 \end{aligned}$$

Expectation of a Linear Function of a Continuous RV

Corollary

If a and b are constants, then

$$E[aX + b] = aE[X] + b$$

Proof

$$\begin{aligned} E[aX+b] &= \int_{-\infty}^{\infty} (ax+b) f(x)\ dx \\ &= a\int_{-\infty}^{\infty} x f(x)\ dx + b\int_{-\infty}^{\infty} f(x)\ dx \\ &= aE[X] + b \end{aligned}$$

Variance of a Continuous RV

Definition

if X is a RV variable with expected value μ, then the variance of X is defined (for any type of RV) by

$$\begin{aligned} \mathrm{Var}(X) &= E\left[(X-\mu)^2\right] \\ &= E[X^2] - \left(E[X]\right)^2 \end{aligned}$$

Example

Find $\mathrm{Var}(X)$ when the pdf of X is: $f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\ dx = \int_0^1 2x^3\ dx = \frac{1}{2}$$

$$\mathrm{Var}(X) = E[X^2] - \left(E[X]\right)^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$$

Common Continuous Probability Distributions

- Uniform Distribution
- Exponential Distribution
- Gamma Distribution
- Normal Distribution

Uniform Distribution

The Standard Uniform Random Variable

A RV is said to be *uniformly* distributed over the interval $(0, 1)$ if its probability density function is given by

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_0^1 1\ dx = 1$$

The probability that X is in any particular subinterval of (0, 1) equals the length of that subinterval. For any $0 < a < b < 1$,

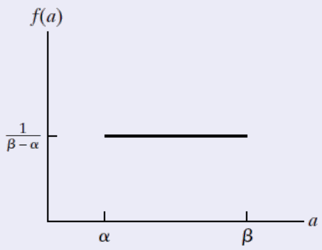
$$P\{a \le X \le b\} = \int_a^b f(x)\ dx = b - a$$

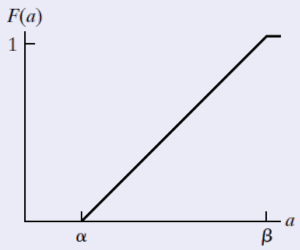
Uniform Distribution

The Uniform Random Variable

A RV is said to be *uniformly* distributed on the interval (α, β) if its pdf is given by

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases} \qquad F(a) = \begin{cases} 0 & a \le \alpha \\ \frac{a-\alpha}{\beta-\alpha} & \alpha < a < \beta \\ 1 & a \ge \beta \end{cases}$$





Uniform Distribution

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f(x)\ dx \\ &= \int_{\alpha}^{\beta} \frac{x}{\beta-\alpha}\ dx \\ &= \frac{1}{\beta-\alpha}\left(\frac{x^2}{2}\bigg|_{\alpha}^{\beta}\right) \\ &= \frac{\beta^2-\alpha^2}{2(\beta-\alpha)} \\ &= \frac{\beta+\alpha}{2} \end{aligned}$$

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x)\ dx \\ &= \int_{\alpha}^{\beta} \frac{x^2}{\beta-\alpha}\ dx \\ &= \frac{\beta^3-\alpha^3}{3(\beta-\alpha)} \\ &= \frac{\beta^2+\alpha\beta+\alpha^2}{3} \\ \mathrm{Var}(X) &= E[X^2] - \left(E[X]\right)^2 \\ &= \frac{\beta^2+\alpha\beta+\alpha^2}{3} - \frac{(\beta+\alpha)^2}{4} \\ &= \frac{(\beta-\alpha)^2}{12} \end{aligned}$$

Uniform Distribution

Example

Buses arrive at a specified stop at 15-minute intervals starting at 7 A.M. That is, they arrive at 7, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that he waits

1. less than 5 minutes for a bus;
2. more than 10 minutes for a bus.

- Let X denote the number of minutes past 7 that the passenger arrives at the stop
- $X \sim \mathrm{Uniform}(0, 30)$

1. $P\{10 < X < 15\} + P\{25 < X < 30\} = \int_{10}^{15} \frac{1}{30}\ dx + \int_{25}^{30} \frac{1}{30}\ dx = \frac{1}{3}$
2. $P\{0 < X < 5\} + P\{15 < X < 20\} = \int_{0}^{5} \frac{1}{30}\ dx + \int_{15}^{20} \frac{1}{30}\ dx = \frac{1}{3}$

Exponential Distribution

The Exponential Random Variable

A continuous RV is said to be an *exponential random variable* with parameter λ if, for some $\lambda > 0$,

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\ dx &= \int_{0}^{\infty} \lambda e^{-\lambda x}\ dx \\ &= -e^{-\lambda x}\Big|_{0}^{\infty} \\ &= 1 \end{aligned}$$

$$\begin{aligned} F(a) &= \int_{-\infty}^{a} f(x)\ dx \\ &= \int_{0}^{a} \lambda e^{-\lambda x}\ dx \\ &= -e^{-\lambda x}\Big|_{0}^{a} \\ &= 1 - e^{-\lambda a} \qquad (a \ge 0) \end{aligned}$$

Exponential Distribution

$$\begin{aligned} E[X^n] &= \int_{-\infty}^{\infty} x^n f(x)\ dx \\ &= \int_{0}^{\infty} x^n \lambda e^{-\lambda x}\ dx \end{aligned}$$

Integrating by parts (with $u = x^n$ and $v = -e^{-\lambda x}$) yields,

$$\begin{aligned} E[X^n] &= -x^n e^{-\lambda x}\Big|_{0}^{\infty} + \int_{0}^{\infty} e^{-\lambda x} n x^{n-1}\ dx \\ &= 0 + \frac{n}{\lambda}\int_{0}^{\infty} \lambda e^{-\lambda x} x^{n-1}\ dx \\ &= \frac{n}{\lambda} E[X^{n-1}] \end{aligned}$$

$$E[X] = \frac{1}{\lambda}E[X^0] = \frac{1}{\lambda} \qquad E[X^2] = \frac{2}{\lambda}E[X] = \frac{2}{\lambda^2}$$

$$\mathrm{Var}(X) = E[X^2] - \left(E[X]\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Exponential Distribution

Examples of RVs that generally obey the exponential distribution

In practice, the exponential distribution often arises as the distribution of the amount of time until some specific event occurs.

- The amount of time (starting from now) until an earthquake occurs
- The amount of time (starting from now) until a new war breaks out
- The amount of time (starting from now) until a telephone call you receive turns out to be a wrong number
- ...

Exponential Distribution

Example

Suppose that the length of a phone call in minutes is an exponential RV with parameter $\lambda = \frac{1}{10}$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait

1. more than 10 minutes;
2. between 10 and 20 minutes.

- Let X denote the length of the call made by the person in the booth
- $X \sim \text{Exponential}(0.1)$; i.e., $F(a) = 1 - e^{-0.1a}$

1. $$P\{X > 10\} = 1 - F(10) = e^{-1} \approx .368$$
2. $$P\{10 < X < 20\} = F(20) - F(10) = e^{-1} - e^{-2} \approx .233$$

Exponential Distribution

Memoryless random variable

We say that a nonnegative RV X is *memoryless* if

$$P\{X > s+t | X > t\} = P\{X > s\} \qquad \forall s,t \geq 0$$

Thinking of X as being the lifetime of some instrument, if the instrument is alive at age t, the distribution of the remaining amount of time that it survives is the same as the original lifetime distribution.

It is as if the instrument does not "remember" that it has already been in use for a time t.

$$\frac{P\{X > s+t, X > t\}}{P\{X > t\}} = P\{X > s\}$$

$$P\{X > s+t\} = P\{X > s\}P\{X > t\}$$

For an exponential RV,

$$P\{X > a\} = 1 - F(a) = e^{-\lambda a}$$

$$e^{-\lambda(s+t)} = e^{-\lambda s}e^{-\lambda t}$$

Exponential Distribution

Example

Suppose that the number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10,000 miles. If a person desires to take a 5000-mile trip, what is the probability that she will be able to complete the trip without having to replace the car battery? What if the distribution is not exponential?

From the memoryless property of the exponential distribution, the remaining lifetime (in thousands of miles) of the battery is exponential with parameter $\lambda = \frac{1}{10}$

$$P\{\text{remaining lifetime} > 5\} = 1 - F(5) = e^{-5\lambda} = e^{-1/2} \approx .604$$

If the lifetime distribution F is not exponential

$$P\{\text{lifetime} > t+5 | \text{lifetime} > t\} = \frac{1 - F(t+5)}{1 - F(t)}$$

Gamma Distribution

The Gamma Random Variable

A RV is said to have a gamma distribution with parameters (α, λ), $\alpha > 0$, $\lambda > 0$, if its pdf is given by

$$f(x) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $\Gamma(\alpha)$, called the gamma function, is defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1}\, dy$$

The gamma distribution with $\lambda = \frac{1}{2}$ and $\alpha = n/2$, $n \in \mathbb{Z}^+$, is called the χ^2_n (read "chi-squared") distribution with n degrees of freedom. The chi-squared distribution arises in practice as the distribution of the error involved in attempting to hit a target in n-dimensional space when each coordinate error is normally distributed.

Gamma Distribution

The Gamma function

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1}\, dy$$

Integration of $\Gamma(\alpha)$ by parts ($u = y^{\alpha-1}$, $v = -e^{-y}$) yields

$$\begin{aligned}\Gamma(\alpha) &= -e^{-y}y^{\alpha-1}\Big|_0^\infty + \int_0^\infty e^{-y}(\alpha-1)y^{\alpha-2}\, dy \\ &= (\alpha-1)\int_0^\infty e^{-y}y^{\alpha-2}\, dy \\ &= (\alpha-1)\Gamma(\alpha-1)\end{aligned}$$

For integral values of α, say, $\alpha = n$,

$$\begin{aligned}\Gamma(n) &= (n-1)\Gamma(n-1) \\ &= (n-1)(n-2)\Gamma(n-2) \\ &= \cdots \\ &= (n-1)(n-2)\cdots 2 \times 1 \times \Gamma(1) \qquad \Longrightarrow \Gamma(n) = (n-1)!\end{aligned}$$

Gamma Distribution

$$\begin{aligned}E[X] &= \int_{-\infty}^\infty xf(x)\, dx \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty \lambda x e^{-\lambda x}(\lambda x)^{\alpha-1}\, dx \\ &= \frac{1}{\lambda\Gamma(\alpha)}\int_0^\infty e^{-\lambda x}(\lambda x)^\alpha\, d\lambda x \\ &= \frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)} \\ &= \frac{\alpha}{\lambda}\end{aligned}$$

$$\begin{aligned}E[X^2] &= \int_{-\infty}^\infty x^2 f(x)\, dx \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty \lambda x^2 e^{-\lambda x}(\lambda x)^{\alpha-1}\, dx \\ &= \frac{1}{\lambda^2\Gamma(\alpha)}\int_0^\infty e^{-\lambda x}(\lambda x)^{\alpha+1}\, d\lambda x \\ &= \frac{\Gamma(\alpha+2)}{\lambda^2\Gamma(\alpha)} \\ &= \frac{\alpha(\alpha+1)}{\lambda^2}\end{aligned}$$

$$\begin{aligned}\text{Var}(X) &= E[X^2] - \Big(E[X]\Big)^2 \\ &= \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} \\ &= \frac{\alpha}{\lambda^2}\end{aligned}$$

Gamma Distribution

The Erlang distribution is a special case of the gamma distribution

The Erlang Random Variable

A RV is said to have an Erlang distribution with parameters (n, λ), $\lambda > 0$, $n \in \mathbb{Z}^+$, if its pdf is given by

$$f(t) = \begin{cases} \dfrac{\lambda(\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} f(t)\ dt = \int_0^{\infty} \frac{\lambda(\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!}\ dt = \frac{1}{(n-1)!} \int_0^{\infty} (\lambda t)^{n-1} e^{-\lambda t}\ d\lambda t$$
$$= \frac{\Gamma(n)}{(n-1)!} = \frac{(n-1)!}{(n-1)!} = 1$$

Gamma Distribution

Erlang distribution with parameters (n, λ) arises as the distribution of the amount of time one has to wait until a total of n events has occurred.

- Let $N(t)$ denote the number of events that have occurred by time t
- Let T_n denote the time at which the n^{th} event occurs
- $N(t) \sim \text{Poisson}(\lambda t)$

$$P\{T_n \leq t\} = P\{N(t) \geq n\} = \sum_{j=n}^{\infty} P\{N(t) = j\} = \sum_{j=n}^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!}$$

$$f(t) = \tfrac{d}{dt} P\{T_n \leq t\} = \sum_{j=n}^{\infty} \frac{e^{-\lambda t} j(\lambda t)^{j-1}\lambda}{j!} - \sum_{j=n}^{\infty} \frac{\lambda e^{-\lambda t}(\lambda t)^j}{j!}$$
$$= \sum_{j=n}^{\infty} \frac{\lambda e^{-\lambda t}(\lambda t)^{j-1}}{(j-1)!} - \sum_{j=n}^{\infty} \frac{\lambda e^{-\lambda t}(\lambda t)^j}{j!}$$
$$= \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}$$

Gamma Distribution

Expectation and variance of Erlang as a special case of Gamma

For integral values of α, say, $\alpha = n$,

$$E[X] = \frac{\alpha}{\lambda} = \frac{n}{\lambda}$$
$$\text{Var}(X) = \frac{\alpha}{\lambda^2} = \frac{n}{\lambda^2}$$

Erlang RV as a sum of n independent identical exponential RVs

$$X = X_1 + X_2 + \cdots + X_n \qquad (X \sim \text{Erlang}(n, \lambda), X_i \sim \text{Exponential}(\lambda), \text{indep.} X_i\text{'s})$$

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n] = \frac{1}{\lambda} + \frac{1}{\lambda} + \cdots + \frac{1}{\lambda} = \frac{n}{\lambda}$$

$$\text{Var}(X) = \text{Var}(X_1) + \text{Var}(X_2) + \cdots + \text{Var}(X_n) = \frac{1}{\lambda^2} + \frac{1}{\lambda^2} + \cdots + \frac{1}{\lambda^2} = \frac{n}{\lambda^2}$$

Normal Distribution

The Normal Random Variable

We say that X is a normal RV, or simply that X is normally distributed, with parameters μ and σ^2 if the density of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty < x < \infty$$

This density function is a bell-shaped curve that is symmetric about μ.

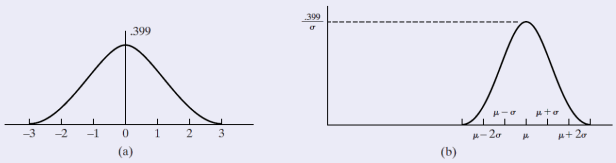


Figure: Normal density function: (a) $\mu = 0, \sigma = 1$; (b) arbitrary μ, σ^2.

Normal Distribution

$f(x)$ is a probability function

$$\int_{-\infty}^{\infty} f(x)\ dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\ dx$$

Making the substitution $y = (x - \mu)/\sigma$,

$$\int_{-\infty}^{\infty} f(x)\ dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\ dy = \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} = 1$$

Normal Distribution

$$I = \int_{-\infty}^{\infty} e^{-y^2/2}\ dy$$
$$I^2 = \int_{-\infty}^{\infty} e^{-y^2/2}\ dy \int_{-\infty}^{\infty} e^{-x^2/2}\ dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(y^2+x^2)/2}\ dy\,dx$$

Changing the variables to polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$ and $dy\,dx = r\,d\theta\,dr$)

$$I^2 = \int_0^{\infty}\int_0^{2\pi} e^{-r^2/2}\ r\ d\theta\,dr = 2\pi \int_0^{\infty} re^{-r^2/2}\ dr = -2\pi e^{-r^2/2}\Big|_0^{\infty} = 2\pi \qquad \Longrightarrow I = \sqrt{2\pi}$$

Normal Distribution

Linear function of normal RVs

if X is normally distributed with parameters μ and σ^2, then $Y = aX + b$ is normally distributed with parameters $a\mu + b$ and $a^2\sigma^2$.

Proof when $a > 0$: (Similar proof when $a < 0$)

$$F_Y(x) = P\{Y \le x\} = P\{aX + b \le x\} = P\left\{X \le \frac{x-b}{a}\right\} = F_X\left(\frac{x-b}{a}\right)$$

$$f_Y(x) = \tfrac{\mathrm{d}}{\mathrm{d}x} F_Y(x) = \frac{1}{a} f_X\left(\frac{x-b}{a}\right)$$

$$= \frac{1}{\sqrt{2\pi} a\sigma} \exp\left\{-\left(\frac{x-b}{a} - \mu\right)^2 \Big/ 2\sigma^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}(a\sigma)} \exp\left\{-(x - b - a\mu)^2 \Big/ 2(a\sigma)^2\right\}$$

$$\implies Y \sim \text{Normal}\left(a\mu + b, a^2\sigma^2\right)$$

Normal Distribution

Corollary

if X is normally distributed with parameters μ and σ^2, then $Z = (X - \mu)/\sigma$ is normally distributed with parameters 0 and 1. Such a random variable is said to be a *standard*, or a unit, normal random variable.

Normal Distribution

For a standard normal random variable $Z = (X - \mu)/\sigma$,

$$E[Z] = \int_{-\infty}^{\infty} x f_Z(x)\ \mathrm{d}x$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2/2}\ \mathrm{d}x$$

$$= -\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\Big|_{-\infty}^{\infty}$$

$$= 0$$

$$X = \mu + \sigma Z$$

$$E[X] = \mu + \sigma E[Z]$$

$$= \mu$$

$$\text{Var}(Z) = E[Z^2]$$

$$= \int_{-\infty}^{\infty} x^2 f_Z(x)\ \mathrm{d}x$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2}\ \mathrm{d}x$$

Integration by parts ($u = x$, $v = -e^{-x^2/2}$)

$$\text{Var}(Z) = \frac{1}{\sqrt{2\pi}} \left(-x e^{-x^2/2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2}\ \mathrm{d}x \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\ \mathrm{d}x$$

$$= 1$$

$$\text{Var}(X) = \sigma^2 \text{Var}(Z)$$

$$= \sigma^2$$

Normal Distribution

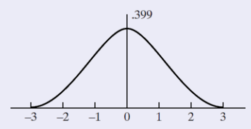
CDF of Normal(0,1)

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\ \mathrm{d}y$$

Symmetry of SN pdf

If $Z \sim \text{Normal}(0, 1)$, then

$$P\{Z \le -x\} = P\{Z > x\}$$



$$\Phi(-x) = 1 - \Phi(x), \quad x \in \mathbb{R}$$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
...										
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990

Table: Area $\Phi(x)$ Under the Standard Normal Curve to the Left of X

Normal Distribution

Answering probability statements about normal RVs

$$F_X(a) = P\{X \le a\}$$

$$= P\left(\frac{X-\mu}{\sigma} \le \frac{a-\mu}{\sigma}\right)$$

$$= P\left(Z \le \frac{a-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Normal Distribution

Example

If $X \sim \text{Normal}(\mu = 3, \sigma^2 = 9)$, find $P\{2 < X < 5\}$.

$$P\{2 < X < 5\} = P\left\{\frac{2-3}{3} < \frac{X-3}{3} < \frac{5-3}{3}\right\}$$

$$= P\left\{-\frac{1}{3} < Z < \frac{2}{3}\right\}$$

$$= \Phi\left(\frac{2}{3}\right) - \Phi\left(-\frac{1}{3}\right)$$

$$= \Phi\left(\frac{2}{3}\right) - \left[1 - \Phi\left(\frac{1}{3}\right)\right]$$

$$\approx \Phi(.67) + \Phi(.33) - 1$$

$$\approx .7486 + .6293 - 1$$

$$= .3779$$

Normal Distribution

The Normal Approximation to the Binomial Distribution

When n is large, a binomial RV variable with parameters n and p will have approximately the same distribution as a normal RV with the same mean and variance as the binomial.

The DeMoivre–Laplace limit theorem

If S_n denotes the number of successes that occur when n independent trials, each resulting in a success with probability p, are performed, then, for any $a < b$,

$$P\left\{a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right\} \to \Phi(b) - \Phi(a) \qquad \text{as } n \to \infty$$

Proof

A special case of the central limit theorem.

Normal Distribution

The normal approximation is quite good when $np(1-p) \ge 10$.

Example

Let X be the number of times that a fair coin that is flipped 40 times lands on heads. Find the probability that $X = 20$.

$$\begin{aligned} P\{X = 20\} &= P\{19.5 \le X < 20.5\} \qquad (\textbf{\textit{continuity correction}}) \\ &= P\left\{\frac{19.5-20}{\sqrt{10}} \le \frac{X-20}{\sqrt{10}} < \frac{20.5-20}{\sqrt{10}}\right\} \\ &\approx P\left\{-.16 \le \frac{X-20}{\sqrt{10}} < .16\right\} \\ &\approx \Phi(.16) - \Phi(-.16) \qquad \approx .1272 \end{aligned}$$

The exact result is

$$P\{X = 20\} = \binom{40}{20}\left(\frac{1}{2}\right)^{40} \approx .1254$$